J. Fluid Mech. (1965), *vol.* 21, *part* 1, *pp.* 173–182
Printed in Great Britain

On the flow of an elastico-viscous liquid in a curved pipe of elliptic cross-section under a pressure-gradient

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(Received 30 May 1964)

Consideration is given to the flow of an elastico-viscous liquid in a curved pipe under a pressure gradient. The cross-section of the pipe is an ellipse, the axes of which are in an arbitrary position with respect to the radius of curvature of the pipe. The method of solution is an extension of that used by Dean (1927) and by Thomas & Walters (1963) in their consideration of flow through a curved pipe of circular cross-section.

It is shown that the liquid elements move along the pipe in two sets of spirals. When the axes of the ellipse are in an asymmetrical position the streamline projections on the cross-section of the pipe are strongly dependent on the elasticity in the liquid. This is not so when the axes are in a symmetrical position. However, in this case, the pitch of the spirals is strongly dependent upon the elasticity of the liquid.

It is also shown that the flux through the pipe is independent of the curvature of the pipe to first order in the curvature.

1. Introduction

In an earlier paper (Thomas & Walters 1963) consideration was given to the flow of an idealized elastico-viscous liquid in a curved pipe of circular cross-section under a pressure gradient. The work was suggested by Dean's treatment of the associated viscous-flow problem (Dean 1927, 1928). It was shown that the main effect of elasticity of the type considered was to decrease the curvature of the streamlines in the central plane of the pipe and also to increase the volume of fluid flowing through the pipe in unit time. The particular elastico-viscous liquid considered in the investigation was that designated liquid B' by Walters (1964), with equations of state†

$$p_{ik} = -pg_{ik} + p'_{ik}, \tag{1}$$

$$p'^{ik}(x,t) = 2\int_{-\infty}^{t} \Psi(t-t') \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^r} e^{(1)mr}(x',t')\,dt', \tag{2}$$

where p_{ik} is the stress tensor, p an arbitrary isotropic pressure, g_{ik} the metric tensor of a fixed co-ordinate system x^i, $e^{(1)}_{ik}$ the rate-of-strain tensor, and

$$\Psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau} e^{-(t-t')/\tau}\,d\tau. \tag{3}$$

† Covariant suffixes are written below, contravariant suffixes above, and the usual summation convention for repeated suffixes is assumed.

In these equations $N(\tau)$ is the relaxation spectrum (Walters 1960) and x'^i $(=x'^i(x,t,t'))$ is the position at time t' of the element that is instantaneously at the point x^i at time t. The liquid designated liquid B by Oldroyd (1950) is a special case of liquid B' obtained by writing†

$$N(\tau) = \eta_0(\lambda_2/\lambda_1)\,\delta(\tau)+\eta_0\{(\lambda_1-\lambda_2)/\lambda_1\}\,\delta(\tau-\lambda_1) \tag{4}$$

in equations (2) and (3). The Newtonian liquid of constant viscosity η_0 is given by

$$N(\tau) = \eta_0\,\delta(\tau). \tag{5}$$

In the present paper, we consider the more general problem of the flow induced by a pressure gradient in a curved pipe of *elliptic* cross-section. The effect of the ellipse being in an asymmetrical position with respect to the radius of curvature of the pipe is also investigated. So far as the authors are aware, the corresponding problem in viscous-flow theory has not been considered.

2. Flow through a curved pipe

The co-ordinate system to be used is shown in figure 1. OS is the axis of the anchor ring formed by the pipe wall. C is the centre of the section of the pipe by a plane through OS making an angle θ with a fixed axial plane. CO is the perpendicular drawn from C on to OS and is of length R. The plane through O

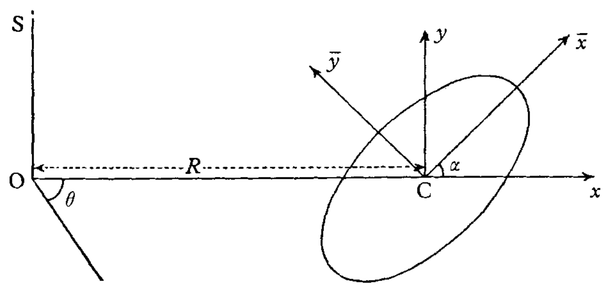

FIGURE 1. The co-ordinate system.

perpendicular to OS and the line traced out by C will be called the central plane and the central line, respectively, of the pipe. Cartesian axes Cx and Cy are drawn in the section of the pipe, Cx being parallel to OC. The equation of the boundary of the pipe, referred to Cartesian axes C$\bar{x}$ and C$\bar{y}$ (which are inclined to the Cx and Cy axes as shown) is taken to be

$$(\bar{x}/a)^2+(\bar{y}/b)^2 = 1.$$

For the sake of mathematical convenience (cf. Dean 1927, 1928; Thomas & Walters 1963) we shall restrict attention to the case when the radius R is large in comparison with the dimensions of the elliptic cross-section of the pipe. The

† δ denotes a Dirac delta function defined in such a way that

$$\delta(x) = 0, \quad (x \neq 0), \quad \int_{-\infty}^{\infty} \delta(x)\,dx = \int_{0}^{\infty} \delta(x)\,dx = 1.$$

equations of motion and the equation of continuity can then be written in the forms†

$$\rho\left[U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}-\frac{W^2}{R}\right]=-\frac{\partial p}{\partial x}+\frac{\partial p'_{(xx)}}{\partial x}+\frac{\partial p'_{(yx)}}{\partial y}-\frac{p'_{(\theta\theta)}}{R}, \tag{6}$$

$$\rho\left[U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}\right]=-\frac{\partial p}{\partial y}+\frac{\partial p'_{(xy)}}{\partial x}+\frac{\partial p'_{(yy)}}{\partial y}, \tag{7}$$

$$\rho\left[U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}\right]=-\frac{1}{R}\frac{\partial p}{\partial \theta}+\frac{\partial p'_{(x\theta)}}{\partial x}+\frac{\partial p'_{(y\theta)}}{\partial y}, \tag{8}$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0, \tag{9}$$

where U, V and W are the physical components of the velocity vector in the x, y and θ directions, respectively, and ρ is the density. In equations (6)–(9) it has been assumed that the motion is steady and that U, V and W (but not p) are independent of θ.

Equations (6)–(9) can be written in non-dimensional form by using the following substitutions:

$$\left.\begin{aligned}&U=\nu u/a,\quad V=\nu v/a,\quad W=W_0 w,\quad x=ax_1,\quad y=ay_1,\\ &p=\eta_0\nu p^*/a^2,\quad J=(-a^2/\eta_0W_0R)\,\partial p/\partial\theta,\\ &p'_{(ik)}=\frac{\eta_0\nu}{a^2}\begin{bmatrix}p''_{(x_1x_1)}, & p''_{(x_1y_1)} & (W_0a/\nu)\,p''_{(x_1\theta)}\\ p''_{(x_1y_1)}, & p''_{(y_1y_1)} & (W_0a/\nu)\,p''_{(y_1\theta)}\\ (W_0a/\nu)\,p''_{(x_1\theta)}, & (W_0a/\nu)\,p''_{(y_1\theta)}, & (W_0a/\nu)^2\,p''_{(\theta\theta)}\end{bmatrix},\end{aligned}\right\} \tag{10}$$

where W_0 has the dimensions of a velocity and

$$\nu=\eta_0/\rho,\quad \eta_0\left(=\int_0^\infty N(\tau)\,d\tau\right)$$

being the limiting viscosity at small rates of shear (Walters 1960). Equations (6)–(9) then become

$$u\frac{\partial u}{\partial x_1}+v\frac{\partial u}{\partial y_1}-\tfrac{1}{2}Lw^2=-\frac{\partial p^*}{\partial x_1}+\frac{\partial p''_{(x_1x_1)}}{\partial x_1}+\frac{\partial p''_{(x_1y_1)}}{\partial y_1}-\tfrac{1}{2}Lp''_{(\theta\theta)}, \tag{11}$$

$$u\frac{\partial v}{\partial x_1}+v\frac{\partial v}{\partial y_1}=-\frac{\partial p^*}{\partial y_1}+\frac{\partial p''_{(x_1y_1)}}{\partial x_1}+\frac{\partial p''_{(y_1y_1)}}{\partial y_1}, \tag{12}$$

$$u\frac{\partial w}{\partial x_1}+v\frac{\partial w}{\partial y_1}=J+\frac{\partial p''_{(x_1\theta)}}{\partial x_1}+\frac{\partial p''_{(y_1\theta)}}{\partial y_1}, \tag{13}$$

$$\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y_1}=0, \tag{14}$$

where it has been convenient to define (cf. Dean 1927, 1928; Thomas & Walters 1963)

$$L=2W_0^2a^3/\nu^2R.$$

The boundary conditions to be associated with equations (2), (11)–(14) are $u=v=w=0$ on $Ax_1^2+By_1^2+2Dx_1y_1=1$, where

$$A=\cos^2\alpha+c^2\sin^2\alpha,\quad B=\sin^2\alpha+c^2\cos^2\alpha,$$
$$D=(1-c^2)\sin\alpha\cos\alpha,\quad\text{and}\quad c^2=a^2/b^2.$$

† Brackets placed round suffixes will be used throughout to denote *physical* components of tensors.

The method of solution given below is one of successive approximation in which it is assumed that u, v and w can be expanded in ascending powers of L.

When the pipe is straight ($L = 0$) $u = v = 0$ and (13) reduces to

$$\nabla_1^2 w = -J, \tag{15}$$

where $\nabla_1^2 = \partial^2/\partial x_1^2 + \partial^2/\partial y_1^2$. The solution of this equation satisfying the boundary conditions on w is

$$w = 1 - Ax_1^2 - By_1^2 - 2Dx_1 y_1, \tag{16}$$

provided $J = 2 + 2c^2$. The corresponding stress distribution is

$$\left.\begin{aligned} p''_{(x_1x_1)} &= 0, \quad p''_{(x_1y_1)} = 0, \quad p''_{(y_1y_1)} = 0, \\ p''_{(\theta x_1)} &= -(2Ax_1 + 2Dy_1), \quad p''_{(\theta y_1)} = -(2By_1 + 2Dx_1), \\ p''_{(\theta\theta)} &= 8m[(Ax_1 + Dy_1)^2 + (By_1 + Dx_1)^2], \end{aligned}\right\} \tag{17}$$

where $$m = K_0/\rho a^2 \text{ and } K_0 = \int_0^\infty \tau N(\tau)\, d\tau.$$

When the pipe is curved and L is sufficiently small we assume that

$$\left.\begin{aligned} u &= Lu_1 + L^2 u_2 + \dots, \\ v &= Lv_1 + L^2 v_2 + \dots, \\ w &= [1 - Ax_1^2 - By_1^2 - 2Dx_1 y_1] + Lw_1 + L^2 w_2 + \dots, \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} p''_{(x_1x_1)} &= Lp''_{(x_1x_1)_1} + L^2 p''_{(x_1x_1)_2} + \dots, \\ p''_{(x_1y_1)} &= Lp''_{(x_1y_1)_1} + L^2 p''_{(x_1y_1)_2} + \dots, \\ p''_{(y_1y_1)} &= Lp''_{(y_1y_1)_1} + L^2 p''_{(y_1y_1)_2} + \dots, \\ p''_{(x_1\theta)} &= -2(Ax_1 + Dy_1) + Lp''_{(x_1\theta)_1} + L^2 p''_{(x_1\theta)_2} + \dots, \\ p''_{(y_1\theta)} &= -2(By_1 + Dx_1) + Lp''_{(y_1\theta)_1} + L^2 p''_{(y_1\theta)_2} + \dots, \\ p''_{(\theta\theta)} &= 8m[(Ax_1 + Dy_1)^2 + (By_1 + Dx_1)^2] + Lp''_{(\theta\theta)_1} + L^2 p''_{(\theta\theta)_2} + \dots. \end{aligned}\right\} \tag{19}$$

In the following we shall work to first order in L.

The equations of state (2) have to be used to determine the relation between the velocity distribution (18) and the stress distribution (19). Initially we shall work in terms of the original variables, using the substitutions (10) later in the analysis.

The displacement functions x'^i corresponding to the velocity distribution (18) are (cf. Thomas & Walters 1963)

$$\left.\begin{aligned} x' &= x - L\nu(t-t')\,u_1/a, \\ y' &= y - L\nu(t-t')\,v_1/a \\ \theta' &= \theta - \frac{(t-t')\,w_0}{R} + \frac{L}{R}\left[-W_0(t-t')\,w_1 + \frac{\nu(t-t')^2}{2a}\left[u_1\frac{\partial w_0}{\partial x} + v_1\frac{\partial w_0}{\partial y}\right]\right], \end{aligned}\right\} \tag{20}$$

where $$w_0 = W_0[1 - (x\cos\alpha + y\sin\alpha)^2/a^2 + (-x\sin\alpha + y\cos\alpha)^2/b^2].$$

The rate-of-strain components $e^{(1)mr}(x', y', t')$ occurring in the equations of state (2) are obtained by writing down the rate-of-strain components for the element at (x, y, θ) at time t, replacing x, y, θ, t by x', y', θ', t', respectively, and using (20).

In this way we obtain†

$$\left.\begin{aligned} e^{(1)xx}(x',y',t') &= e^{(1)xx}(x,y,t,t') = (L\nu/a)\,\partial u_1/\partial x,\\ e^{(1)yy}(x',y',t') &= e^{(1)yy}(x,y,t,t') = (L\nu/a)\,\partial v_1/\partial y,\\ e^{(1)xy}(x',y',t') &= e^{(1)xy}(x,y,t,t') = \frac{L\nu}{2a}\left[\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right],\\ e^{(1)\theta x}(x',y',t') &= e^{(1)\theta x}(x,y,t,t')\\ &= \frac{1}{2R}\left\{\frac{\partial w_0}{\partial x}+L\left[W_0\frac{\partial w_1}{\partial x}-(t-t')\frac{\nu}{a}\left[u_1\frac{\partial^2 w_0}{\partial x^2}+v_1\frac{\partial^2 w_0}{\partial x\,\partial y}\right]\right]\right\},\\ e^{(1)\theta y}(x',y',t') &= e^{(1)\theta y}(x,y,t,t')\\ &= \frac{1}{2R}\left\{\frac{\partial w_0}{\partial y}+L\left[W_0\frac{\partial w_1}{\partial y}-(t-t')\frac{\nu}{a}\left[u_1\frac{\partial^2 w_0}{\partial x\,\partial y}+v_1\frac{\partial^2 w_0}{\partial y^2}\right]\right]\right\}. \end{aligned}\right\} \quad (21)$$

Equations (20) and (21) can now be used to determine the physical components of the partial-stress tensor. After some reduction and use of (10) we obtain

$$\left.\begin{aligned} &p''_{(x_1x_1)_1} = 2\,\partial u_1/\partial x_1,\quad p''_{(y_1y_1)_1} = 2\,\partial v_1/\partial y,\quad p''_{(x_1y_1)_1} = \partial u_1/\partial y_1+\partial v_1/\partial x_1,\\ &p''_{(\theta x)_1} = \partial w_1/\partial x_1+m[-6(Ax_1+Dy_1)\,\partial u_1/\partial x_1-4(By_1+Dx_1)\,\partial u_1/\partial y_1\\ &\qquad -2(By_1+Dx_1)\,\partial v_1/\partial x_1+2Au_1+2Dv_1],\\ &p''_{(\theta y_1)_1} = \partial w_1/\partial y_1+m[-6(By_1+Dx_1)\,\partial v_1/\partial y_1-4(Ax_1+Dy_1)\,\partial v_1/\partial x_1\\ &\qquad -2(Ax_1+Dy_1)\,\partial u_1/\partial y_1+2Du_1+2Bv_1],\\ &p''_{(\theta\theta)_1} = -8m[(Ax_1+Dy_1)\,\partial w_1/\partial x_1+(By_1+Dx_1)\,\partial w_1/\partial y_1]\\ &\quad +24s[(Ax_1+Dy_1)^2\,\partial u_1/\partial x_1+(By_1+Dx_1)^2\,\partial v_1/\partial y_1\\ &\quad +(Ax_1+Dy_1)(By_1+Dx_1)[\partial v_1/\partial x_1+\partial u_1/\partial y_1]\\ &\quad -u_1[A(Ax_1+Dy_1)+D(By_1+Dx_1)]-v_1[B(By_1+Dx_1)+D(Ax_1+Dy_1)]], \end{aligned}\right\} \quad (22)$$

where $$s = \frac{\eta_0 S_0}{\rho^2 a^4} \quad \text{and} \quad S_0 = \int_0^\infty \tau^2 N(\tau)\,d\tau.$$

It is convenient at this stage, after inspection of (9), to introduce a stream function $\chi(x,y)$ defined by

$$U = -\,\partial\chi/\partial y,\quad V = \partial\chi/\partial x. \quad (23)$$

Writing $$\chi = \nu[L\chi_1+L^2\chi_2+\ldots,],$$

we have $$u_1 = -\,\partial\chi_1/\partial y_1,\quad v_1 = \partial\chi_1/\partial x_1,\ \text{etc.} \quad (24)$$

Substituting (22) and (24) into (11) and (12) and eliminating p^*, we obtain (on equating coefficients of L)

$$\nabla_1^4\chi_1 = -2[1-Ax_1^2-By_1^2-2Dx_1y_1][By_1+Dx_1] -8m[(D^2+B^2)y_1+D(A+B)x_1]. \quad (25)$$

The solution of (25) satisfying the boundary conditions

$$\partial\chi_1/\partial x_1 = \partial\chi_1/\partial y_1 = 0 \quad \text{on} \quad Ax_1^2+By_1^2+2Dx_1y_1 = 1$$

is $$\chi_1 = [1-Ax_1^2-By_1^2-2Dx_1y_1]^2\,[dy_1^3+ex_1^2y_1+fy_1+gx_1+hx_1y_1^2+jx_1^3], \quad (26)$$

† $e^{(1)\theta\theta}$ is neglected because it is at most of order a/R and is seen to occur only in the expression for $p'_{(\theta\theta)}$ which is itself divided by R in the stress equations of motion.

where $d-j$ are complicated algebraic functions of c, α and m. When $\alpha = 0$, the formula for χ_1 is simplified. In this case $g = h = j = 0$ and

$$\left.\begin{aligned} d &= \Delta^{-1}[39c^2+26c^6+15c^8], \\ e &= \Delta^{-1}[3c^2+2c^4+75c^6], \\ f &= -[39c^2+(212+180m)c^4+(1114+1200m)c^6+(820+6840m)c^8 \\ &\qquad +(375+5040m)c^{10}+2100mc^{12}]/\Delta[1+2c^2+5c^4], \end{aligned}\right\} \tag{27}$$

where $\Delta = 180[3+20c^2+114c^4+84c^6+35c^8]$. Substitution from (22) into (13) and consideration of only those terms involving L gives

$$\Delta_1^2[w_1+2m[(Ax_1+Dy_1)\,\partial\chi_1/\partial y_1-(By_1+Dx_1)\,\partial\chi_1/\partial x_1]] \\ = 2[(Ax_1+Dy_1)\,\partial\chi_1/\partial y_1-(By_1+Dx_1)\,\partial\chi_1/\partial x_1]. \tag{28}$$

The associated boundary conditions are $w_1 = 0$ on $Ax_1^2+By_1^2+2Dx_1y_1 = 1$ and w_1 is finite at $x_1 = y_1 = 0$. After substitution from (26), the solution of equation (28) is found to be of the form

$$\begin{aligned} w_1 = (1-Ax_1^2-By_1^2-2Dx_1y_1)\,[&I_1x_1^7+I_2x_1^5+I_3x_1^3+I_4x_1+I_5x_1y_1^2+I_6x_1^3y_1^2 \\ &+I_7x_1^5y_1^2+I_8x_1y_1^4+I_9x_1^3y_1^4+I_{10}x_1y_1^6+J_1y_1^7+J_2y_1^5+J_3y_1^3+J_4y_1 \\ &+J_5y_1x_1^2+J_6y_1^3x_1^2+J_7y_1^5x_1^2+J_8y_1x_1^4+J_9y_1^3x_1^4+J_{10}y_1x_1^6], \end{aligned} \tag{29}$$

where I_1 to I_{10}, J_1 to J_{10} are algebraic functions of c, α and m whose forms are too complicated to be given here. When $\alpha = 0$, J_1 to J_{10} are all zero.

When $c = 1$, equations (26) and (29) reduce to those given by Thomas & Walters (1963) in their consideration of flow through a curved pipe of circular cross-section.

3. Streamline projections

(i) *In the plane of the pipr.* The streamline projections on the cross-section of the pipe are represented by $\chi_1 = \text{const.}$, where χ_1 is given by (26). Figures 2–7 contain streamline projections for various values of c, α and m. Figures 2, 4 and 6 relate to an elastico-viscous liquid with $m = 1$ and may be compared with figures 3, 5 and 7 for a Newtonian liquid ($m = 0$). It will be observed that the streamline projections are strongly dependent on the elasticity of the liquid when the axes of the ellipse are in an asymmetrical position but are very difficult to distinguish when $\alpha = 0$.

(ii) *In the central plane.* When $\alpha = 0$, the central plane is a plane of symmetry (cf. figures 4, 5) and particles originally in this plane will remain so during the subsequent motion. In this case, an investigation of the streamlines in the central plane can be used to study the pitch of the spirals along which the liquid elements move.

The equation of the streamlines in the central plane can be shown to be (cf. Thomas & Walters 1963)

$$\theta = \frac{1}{4(h^2-1)\,hne}\ln\left[\left(\frac{1+x_1}{1-x_1}\right)^h\left(\frac{h-x_1}{h+x_1}\right)\right], \tag{30}$$

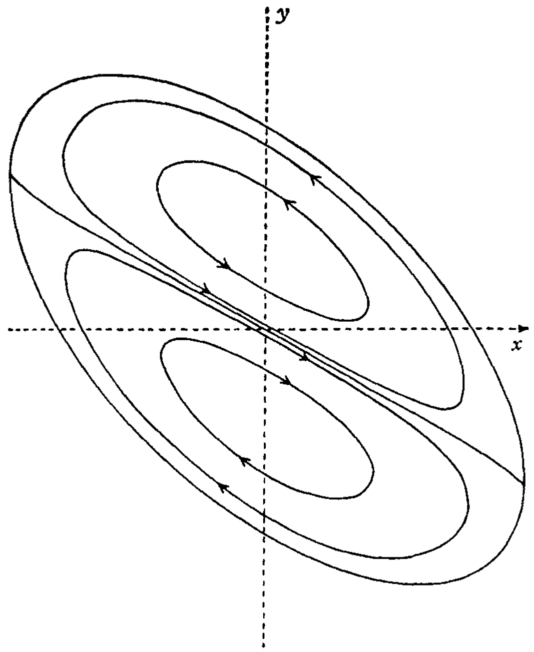

FIGURE 2. Paths of particles projected on the cross-section of the pipe for $c = 0{\cdot}5$, $\alpha = \frac{1}{4}\pi$, $m = 1{\cdot}0$.

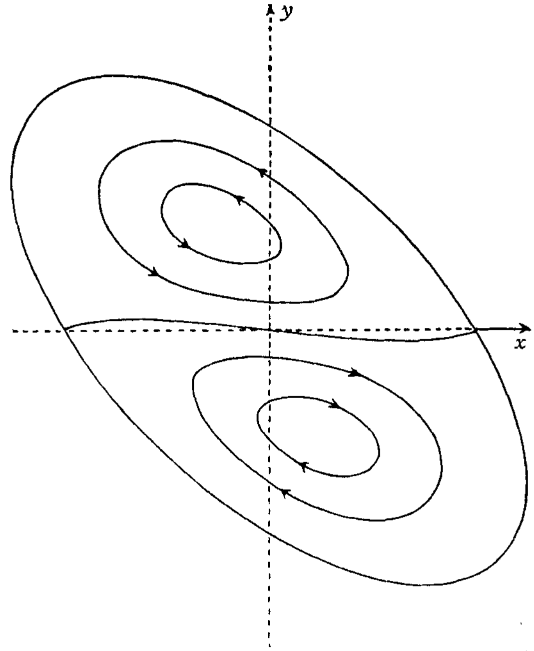

FIGURE 3. Paths of particles projected on the cross-section of the pipe for $c = 0{\cdot}5$, $\alpha = \frac{1}{4}\pi$, $m = 0$.

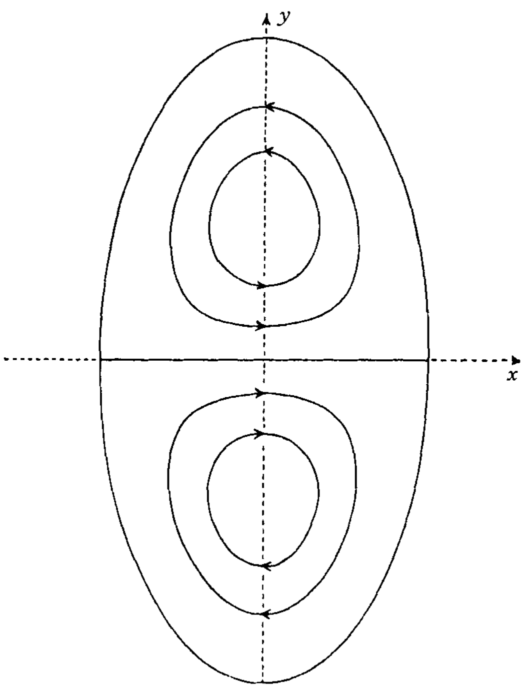

FIGURE 4. Paths of particles projected on the cross-section of the pipe for $c = 0{\cdot}5$, $\alpha = 0$, $m = 1{\cdot}0$.

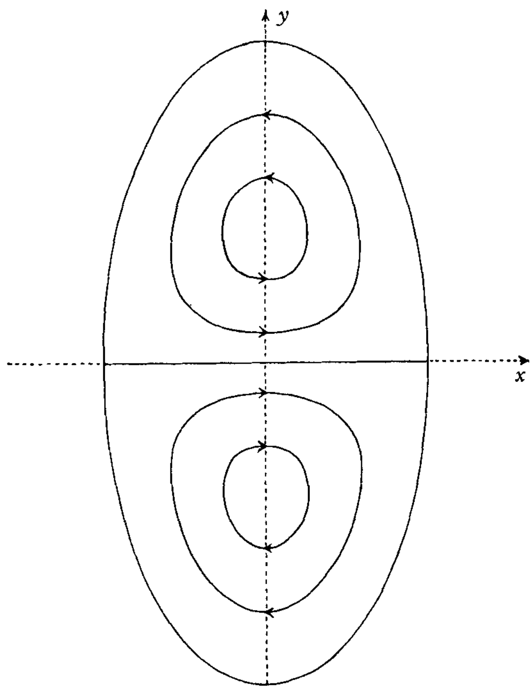

FIGURE 5. Paths of particles projected on the cross-section of the pipe for $c = 0{\cdot}5$, $\alpha = 0$, $m = 0$.

where $h^2 = -f/e$, f and e being given by (27) and n is a Reynolds number defined as $W_0 a\rho/\eta_0$. In equation (30) it has been assumed that θ is measured from the point where the streamline crosses the central line $x_1 = 0$. When $c = 1$, equation (30) reduces to that given by Thomas & Walters (1963) in their consideration of flow through a pipe of circular cross-section.

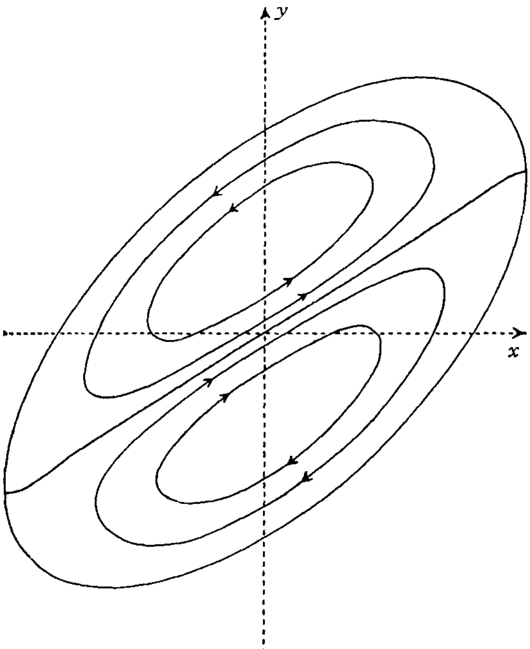

FIGURE 6. Paths of particles projected on the cross-section of the pipe for $c = 2{\cdot}0$, $\alpha = \frac{1}{4}\pi$, $m = 1{\cdot}0$.

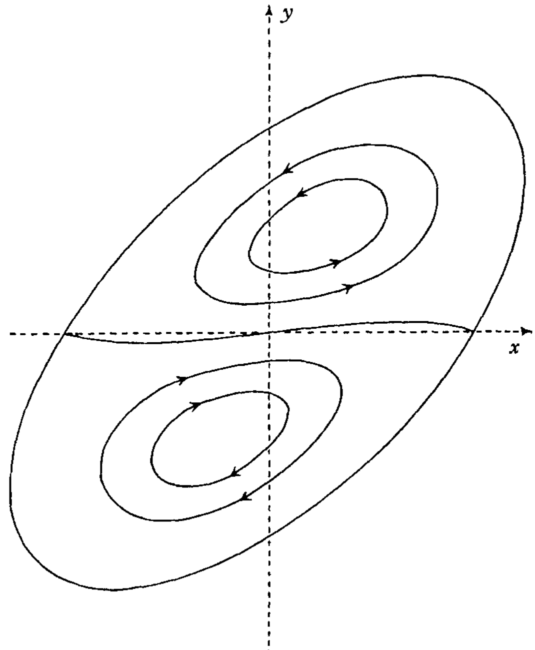

FIGURE 7. Paths of particles projected on the cross-section of the pipe for $c = 2{\cdot}0$, $\alpha = \frac{1}{4}\pi$, $m = 0$.

Figure 8 illustrates the dependence of the form of the streamlines in the central plane upon the parameter m for a particular c, curves being drawn for $m = 0$, $m = 0{\cdot}1$, $m = 1{\cdot}0$ when $c = 2$.† The Reynolds number used in the calculation was 63·3 and for the sake of convenience in drawing we have assumed that a/R is $\frac{1}{2}$ (cf. Thomas & Walters 1963). It is seen that an increase in m leads to a spectacular decrease in the curvature of the streamlines in the central plane.

To illustrate the dependence of the curvature of the streamlines in the central plane upon c we consider the quantity $\theta_{0\cdot 9}$—the value of θ calculated from (30) when $x_1 = 0{\cdot}9$. Curves of $\theta_{0\cdot 9}$ against c are shown in figure 9 for $m = 0$, $m = 0{\cdot}1$ and $m = 1{\cdot}0$, with $n = 63{\cdot}3$. It will be observed that the presence of elasticity of the type considered has a marked effect upon $\theta_{0\cdot 9}$, particularly when $c > 1$. For such values of c, $\theta_{0\cdot 9}$ increases steadily with c in the case of the Newtonian liquid ($m = 0$); on the other hand, when $m > 0$, $\theta_{0\cdot 9}$ tends to constant values as c increases. It is not difficult to show that these constant values are inversely

† The curves for $m = 0{\cdot}1$ and $m = 1{\cdot}0$ have been drawn between $x_1 = -0{\cdot}9$ and $x_1 = 0{\cdot}9$. For the sake of presentation, the curve for $m = 0$ has been drawn between $x_1 = -0{\cdot}9$ and $x_1 = 0{\cdot}8$.

proportional to m. When $c < 1$, $\theta_{0\cdot9}$ increases steadily with decreasing c in all cases. In this region, the variation of $\theta_{0\cdot9}$ with m is less marked and all the curves merge as $c \to 0$.

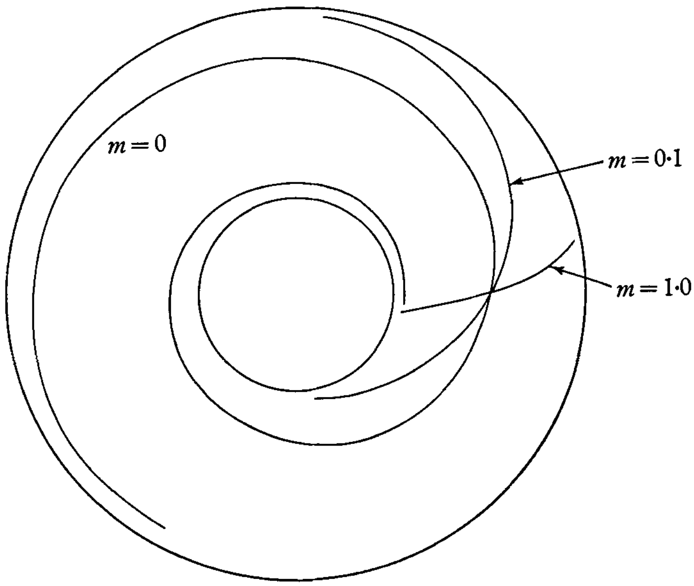

FIGURE 8. The path of a particle in the central plane for $c = 2\cdot0$ and various values of m.

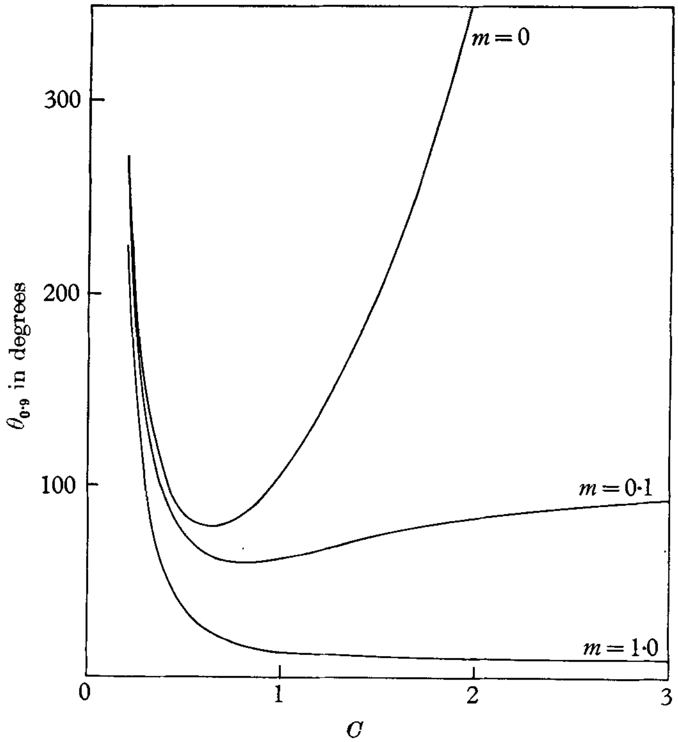

FIGURE 9. Graphs of $\theta_{0\cdot9}$ against c for various values of m.

4. The rate of discharge of liquid through the pipe

The rate of flow through the pipe is a constant times

$$\iint_{\sigma} [1 - Ax_1^2 - By_1^2 - 2Dx_1 y_1 + Lw_1 + \ldots]\, dx_1\, dy_1,$$

where σ is the area bounded by the curve $Ax_1^2 + By_1^2 + 2Dx_1y_1 = 1$ and w_1 is given by (29). When note is taken of the functional form of equation (29), it is not difficult to show that w_1 makes no contribution to the rate of flow through the pipe. Hence the flux through the pipe is independent of the curvature to the first approximation even when the ellipse is asymmetrically placed with respect to the radius of curvature of the pipe.

It has not been possible to consider the variation of flux with L^2 even when $\alpha = 0$. The simple approach used by Dean (1928) and Thomas & Walters (1963) in the case of a curved pipe of circular cross-section is no longer applicable, and the work involved in considering all the relevant second-order terms is prohibitive.

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